

The Dot Product

In this section, we will now concentrate on the vector operation called the dot product. The dot product of two vectors will produce a scalar instead of a vector as in the other operations that we examined in the previous section. The dot product is equal to the sum of the product of the horizontal components and the product of the vertical components.

If $\mathbf{v} = a_1 \mathbf{i} + b_1 \mathbf{j}$ and $\mathbf{w} = a_2 \mathbf{i} + b_2 \mathbf{j}$ are vectors then their dot product is given by:

$$\mathbf{v} \cdot \mathbf{w} = a_1 a_2 + b_1 b_2$$

Properties of the Dot Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors and c is a scalar then:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

$$\mathbf{0} \cdot \mathbf{v} = 0$$

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$$

$$(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$$

Example 1: If $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} - 7\mathbf{j}$ then find $\mathbf{v} \cdot \mathbf{w}$.

Solution:

$$\mathbf{v} \cdot \mathbf{w} = a_1 a_2 + b_1 b_2$$

$$\mathbf{v} \cdot \mathbf{w} = (5)(3) + (2)(-7)$$

$$\mathbf{v} \cdot \mathbf{w} = 15 - 14$$

$$\mathbf{v} \cdot \mathbf{w} = 1$$

Example 2: If $\mathbf{u} = -\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = 7\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} + \mathbf{j}$ then find $(3\mathbf{u}) \cdot (\mathbf{v} + \mathbf{w})$.

Solution:

Find $3\mathbf{u}$

$$3\mathbf{u} = 3(-\mathbf{i} + 3\mathbf{j})$$

$$3\mathbf{u} = -3\mathbf{i} + 9\mathbf{j}$$

Find $\mathbf{v} + \mathbf{w}$

$$\mathbf{v} + \mathbf{w} = (7\mathbf{i} - 4\mathbf{j}) + (2\mathbf{i} + \mathbf{j})$$

$$\mathbf{v} + \mathbf{w} = (7 + 2)\mathbf{i} + (-4 + 1)\mathbf{j}$$

$$\mathbf{v} + \mathbf{w} = 9\mathbf{i} - 3\mathbf{j}$$

Example 2 (Continued):

Find the dot product between $(3\mathbf{u})$ and $(\mathbf{v} + \mathbf{w})$

$$(3\mathbf{u}) \cdot (\mathbf{v} + \mathbf{w}) = (-3\mathbf{i} + 9\mathbf{j}) \cdot (9\mathbf{i} - 3\mathbf{j})$$

$$(3\mathbf{u}) \cdot (\mathbf{v} + \mathbf{w}) = (-3)(9) + (9)(-3)$$

$$(3\mathbf{u}) \cdot (\mathbf{v} + \mathbf{w}) = -27 - 27$$

$$(3\mathbf{u}) \cdot (\mathbf{v} + \mathbf{w}) = -54$$

An alternate formula for the dot product is available by using the angle between the two vectors. If \mathbf{v} and \mathbf{w} are two nonzero vectors and θ is the smallest nonnegative angle between them then their dot product is given:

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

This same equation could be solved for theta if the angle between the vectors needed to be determined.

$$\theta = \cos^{-1} \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

Example 3: If $\mathbf{u} = 6\mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$ then find the angle θ between the vectors. Round the answer to the nearest tenth of a degree, if necessary.

Solution:

Find the magnitude of \mathbf{u}

$$\|\mathbf{u}\| = \sqrt{a^2 + b^2}$$

$$\|\mathbf{u}\| = \sqrt{(6)^2 + (-2)^2}$$

$$\|\mathbf{u}\| = \sqrt{36 + 4}$$

$$\|\mathbf{u}\| = \sqrt{40}$$

$$\|\mathbf{u}\| = 2\sqrt{10}$$

Find the magnitude of \mathbf{v}

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2}$$

$$\|\mathbf{v}\| = \sqrt{(3)^2 + (5)^2}$$

$$\|\mathbf{v}\| = \sqrt{9 + 25}$$

$$\|\mathbf{v}\| = \sqrt{34}$$

Example 3 (Continued):

Find the dot product of \mathbf{u} and \mathbf{v}

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= a_1 a_2 + b_1 b_2 \\ \mathbf{u} \cdot \mathbf{v} &= (6)(3) + (-2)(5) \\ \mathbf{u} \cdot \mathbf{v} &= 18 - 10 \\ \mathbf{u} \cdot \mathbf{v} &= 8\end{aligned}$$

Find the angle between the vectors

$$\begin{aligned}\theta &= \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \\ \theta &= \cos^{-1} \frac{8}{(2\sqrt{10})(\sqrt{34})} \\ \theta &\approx \cos^{-1}(0.2169) \\ \theta &\approx 77.5^\circ\end{aligned}$$

When comparing two lines they were described as being parallel, perpendicular, or neither depending on the values of their slopes. The same can be done with vectors but orthogonal is used instead of the term perpendicular. However, we would be looking at the measure of the angle between the two vectors.

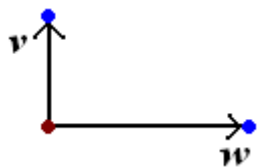
Parallel vectors

Two vectors are parallel when the angle between them is either 0° (the vectors point in the same direction) or 180° (the vectors point in opposite directions) as shown in the figures below.



Orthogonal vectors

Two vectors are orthogonal when the angle between them is a right angle (90°). The dot product of two orthogonal vectors is zero.



Example 4: Determine if $\mathbf{u} = -\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = 9\mathbf{i} - 3\mathbf{j}$ are orthogonal.

Solution:

Verify that the dot product is 0

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (-\mathbf{i} - 3\mathbf{j}) \cdot (9\mathbf{i} - 3\mathbf{j}) \\ \mathbf{u} \cdot \mathbf{v} &= (-1)(9) + (-3)(-3) \\ \mathbf{u} \cdot \mathbf{v} &= -9 + 9 \\ \mathbf{u} \cdot \mathbf{v} &= 0\end{aligned}$$

The dot product is zero so the vectors are orthogonal.

There are real world applications of vectors that will require for the vectors to be broken down into its orthogonal components. By breaking a vector into its orthogonal components we can express a vector as the sum of vectors. The components are formed by what is called “vector projection.” Vector projection involves drawing a line from the terminal point of the vector we want to project down to form a right angle with a line passing through the other vector. If for example vector \mathbf{v} is projected upon vector \mathbf{w} , then the projected vector would be denoted as $\text{proj}_{\mathbf{w}} \mathbf{v}$ and is given by:

$$\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w}$$

The projected vector can then be subtracted from vector \mathbf{v} to finish the decomposition process of \mathbf{v} into its orthogonal components.

Decomposition of \mathbf{v} into vector components

If vectors \mathbf{v} and \mathbf{w} are two nonzero vectors, then vector \mathbf{v} can be expressed as the sum of its orthogonal component vectors \mathbf{v}_1 and \mathbf{v}_2 , where \mathbf{v}_1 (the vector projected onto \mathbf{w}) is parallel to \mathbf{w} and \mathbf{v}_2 is orthogonal to \mathbf{w} .

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w} \quad \text{and} \quad \mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$$

Since the vector components are orthogonal their dot product must be equal to zero. You can use this as a way of checking to make sure the vector components are correct.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$$

Example 5: Let $\mathbf{v} = -5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} - 4\mathbf{j}$. Decompose \mathbf{v} into its vector components \mathbf{v}_1 and \mathbf{v}_2 , where \mathbf{v}_1 is parallel to \mathbf{w} and \mathbf{v}_2 is orthogonal to \mathbf{w} .

Solution:

Find the magnitude of vector \mathbf{w}

$$\|\mathbf{w}\| = \sqrt{a^2 + b^2}$$

$$\|\mathbf{w}\| = \sqrt{(2)^2 + (-4)^2}$$

$$\|\mathbf{w}\| = \sqrt{4 + 16}$$

$$\|\mathbf{w}\| = \sqrt{20}$$

Find the dot product of \mathbf{v} and \mathbf{w}

$$\mathbf{v} \cdot \mathbf{w} = (-5\mathbf{i} + 2\mathbf{j}) \cdot (2\mathbf{i} - 4\mathbf{j})$$

$$\mathbf{v} \cdot \mathbf{w} = (-5)(2) + (2)(-4)$$

$$\mathbf{v} \cdot \mathbf{w} = -10 - 8$$

$$\mathbf{v} \cdot \mathbf{w} = -18$$

Use vector projection to find \mathbf{v}_1

$$\mathbf{v}_1 = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w}$$

$$\mathbf{v}_1 = \frac{-18}{(\sqrt{20})^2} (2\mathbf{i} - 4\mathbf{j})$$

$$\mathbf{v}_1 = -\frac{9}{10} (2\mathbf{i} - 4\mathbf{j})$$

$$\mathbf{v}_1 = -\frac{9}{5} \mathbf{i} + \frac{18}{5} \mathbf{j}$$

Example 5 (Continued):

Use vector subtraction to find \mathbf{v}_2

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$$

$$\mathbf{v}_2 = (-5\mathbf{i} + 2\mathbf{j}) - \left(-\frac{9}{5}\mathbf{i} + \frac{18}{5}\mathbf{j}\right)$$

$$\mathbf{v}_2 = (-5\mathbf{i} + 2\mathbf{j}) + \left(\frac{9}{5}\mathbf{i} - \frac{18}{5}\mathbf{j}\right)$$

$$\mathbf{v}_2 = \left(-5 + \frac{9}{5}\right)\mathbf{i} + \left(2 - \frac{18}{5}\right)\mathbf{j}$$

$$\mathbf{v}_2 = -\frac{16}{5}\mathbf{i} - \frac{8}{5}\mathbf{j}$$

Check of the answer

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \left(-\frac{9}{5}\mathbf{i} + \frac{18}{5}\mathbf{j}\right) \cdot \left(-\frac{16}{5}\mathbf{i} - \frac{8}{5}\mathbf{j}\right)$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \left(-\frac{9}{5}\right)\left(-\frac{16}{5}\right) + \left(\frac{18}{5}\right)\left(-\frac{8}{5}\right)$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \frac{144}{25} - \frac{144}{25}$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$$

Since the dot product is zero, the vector \mathbf{v}_1 would be parallel to \mathbf{w} and \mathbf{v}_2 would be orthogonal to \mathbf{w} .

One example of how the magnitude of a projected vector is used would be in the field of physics when calculating the work done by a force moving an object over a distance. The force applied to the object will not always be applied along the line of motion.



The amount of work W done by a force \mathbf{F} moving an object from point A to point B is given by:

$$W = \mathbf{F} \cdot \overline{\mathbf{AB}}$$

$$W = \|\mathbf{F}\| \|\overline{\mathbf{AB}}\| \cos \theta, \text{ where } \|\overline{\mathbf{AB}}\| \text{ is the distance from A to B.}$$

Example 6: A force of 90 pounds is used to pull an object along a level surface. The force is applied at an angle of 43° to the surface and moves the object a total of 250 feet. How much work was done? Round the answer to the nearest foot-pound.

Solution:

$$W = \|F\| \|\overline{AB}\| \cos \theta$$

$$W = (90 \text{ pounds})(250 \text{ feet}) \cos 43^\circ$$

$$W \approx 16455.4583$$

Rounded off to the nearest foot-pound, the amount of work done is approximately 16,455 foot-pounds.